The Chain Rule

could use other letters, r, q, u, etc

f - function in variable x, $y = f(x)$

(Instantaneous)

$\frac{dy}{dx}$ = Rate of change of y with respect to x. Measures how y changes as x changes.

a function in x

Examples

$$f(x) = 2x + x^2 \Rightarrow \frac{dy}{dx} = 2 + 2x$$

$$f(u) = \sqrt{u} + 3 \Rightarrow \frac{dy}{du} = \frac{1}{2} u^{-\frac{1}{2}}$$

$$f(r) = r^3 \Rightarrow \frac{dy}{dr} = 3r^2$$

Remark: The advantage of $\frac{dy}{dx}$ notation is that it explicitly tells us what the independent variable is. This is especially important when there are several possible independent variables.

Motivating Example

A leaking oil well is spreading a circular film over the water surface.

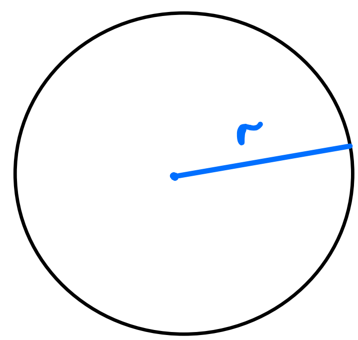
At time t (in minutes) after the start of the leak, the radius of the circular slick is $r(t) = 4t$ (in feet).

1/ What is the rate of change of surface area with respect to the radius?

2/ What is the rate of change the radius with respect to time?

3/ What is the rate of change of the surface area with respect to time?
((Can we relate 3/ to 1/ and 2/?)

1/ A = Area of circle
r = radius of circle
t = time



pi

$$A = \pi r^2$$

Rate of change of A with respect to r $= \dfrac{dA}{dr} = \dfrac{d}{dr}(\pi r^2) = 2\pi r$

2/ $r(t) = 4t$

Rate of change of r with respect to t $= \dfrac{dr}{dt} = \dfrac{d}{dt}(4t) = 4$

composition gives area as a function in t.

3/ $A(r) = \pi r^2$
$r(t) = 4t$ $\Rightarrow$ $A(r(t)) = \pi(4t)^2 = 16\pi t^2$

$\Rightarrow$ Rate of change of A with respect to t $= \dfrac{dA}{dt} = \dfrac{d}{dt}(16\pi t^2) = 32\pi t.$

Observation :

$$32\pi t = 2\pi(4t) \times 4 = 2\pi\, r(t) \cdot 4$$

$$\underset{\dfrac{dA}{dt}}{\shortparallel} \qquad\qquad \underset{\dfrac{dA}{dr}}{\shortparallel} \qquad \underset{\dfrac{dr}{dt}}{\shortparallel}$$

thought of as a function in t

$$\Rightarrow \frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$\Rightarrow$ Rate of change of area with respect to time = Rate of change of area with respect to radius $\times$ Rate of radius with respect to time.

Chain Rule (Form 1)

Let $y = f(u)$ and $u = g(x)$,

then $y = f(g(x))$, and

thought of as a function in x (replace u with $g(x)$)

Not really cancellation of fractions. Just clever notation.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Example $\quad y = (3x^2 - 5x)^{1/2} \Rightarrow \dfrac{dy}{dx} = ?$

Let $y = u^{1/2}$ and $u = 3x^2 - 5x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2} u^{-\frac{1}{2}} \cdot (6x - 5)$$

$$= \frac{1}{2}(3x^2 - 5x)^{-\frac{1}{2}} \cdot (6x-5)$$

replacing u with $3x^2 - 5x$

Example $y = (2x^2 - 1)^{-2} \Rightarrow \frac{dy}{dx} = ?$

Prime Notation : $\frac{dy}{du} = f'(u) = f'(g(x))$

$$\frac{du}{dx} = g'(x)$$

$\Rightarrow$ **Chain Rule** (Form 2)

$$\boxed{\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\, g'(x)}$$

$$\frac{dy}{dx} \qquad\qquad \frac{dy}{du} \qquad \frac{du}{dx}$$

Conclusion : The Chain Rule tells us how to differentiate compositions of functions.

Remark : It's up to you which form you use. They both give same answer.

Hard part : Identifying $f(x)$ and $g(x)$ correctly.

Examples

Could multiply out. It would take ages

1/ $y = (3x+1)^7 \Rightarrow \frac{dy}{dx} = ?$

$y = f(g(x))$ where $g(x) = 3x+1$ and $f(x) = x^7$

$\Rightarrow g'(x) = 3$

$f'(x) = 7x^6$

$$\Rightarrow \frac{dy}{dx} = f'(g(x))\, g'(x) = 7(3x+1)^6 \cdot 3$$

3/ Useful General Fact :

If $g(x) = mx+b$ (linear)

$\Rightarrow g'(x) = m$

derivative of f evaluated at $mx+b$

$$\Rightarrow \boxed{\frac{d}{dx}\left(f(mx+b)\right) = f'(mx+b) \cdot m}$$

remember to multiply by m

2. $y = \dfrac{1}{\sqrt{x^3 - x}} \Rightarrow \dfrac{dy}{dx} = ?$

Could do quotient rule, but first notice

$y = (x^3 - x)^{-\frac{1}{2}}$. $\Rightarrow y = f(g(x))$

where $g(x) = x^3 - x$ and $f(x) = x^{-\frac{1}{2}}$

$\Rightarrow g'(x) = 3x^2 - 1$, $f(x) = -\frac{1}{2} x^{-\frac{3}{2}}$

$$\Rightarrow \frac{dy}{dx} = f'(g(x))\, g'(x)$$

$$= -\frac{1}{2} (x^3 - x)^{-\frac{3}{2}} \cdot (3x^2 - 1)$$